

1 Math 406, Project 2: Due 18 th Dec. in Math 108 at 12 noon

Electrical Impedance Tomography (EIT)

2 Problem description

In this project we illustrate the application of Green's functions to the problem of determining changes of conductivity $\sigma(r, \theta)$ within a circular region from a finite number of voltage measurement along the boundary for a current I applied to parts of the boundary. The boundary value problem is given by

$$\begin{aligned}\nabla \cdot (\sigma \nabla u) &= 0 \text{ on } \Omega = \{(r, \theta) : r \leq a, 0 \leq \theta < 2\pi\} \\ \sigma \frac{\partial u}{\partial n} &= I [\delta(\theta - \alpha) - \delta(\theta + \alpha)]\end{aligned}$$

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This problem is a so-called inverse problem in which material properties are determined from measurements of field quantities. This process can typically lead to images of anomalies that it is important to identify - for example, tumors are known to have significantly higher conductivities than normal tissue thus being able to identify regions with perturbed conductivities can lead to images of tumors. Other applications could be the identification of plastic (low conductivity) landmines from the surrounding soil in which they are buried, or changes in conductivities to identify mineral deposits.

3 Green's function assuming a uniform conductivity

Solve the boundary value problem assuming $\sigma = \text{const}$

$$\begin{aligned}\nabla^2 u &= 0 \text{ on } \Omega = \{(r, \theta) : r \leq a, 0 \leq \theta < 2\pi\} \\ \sigma \frac{\partial u}{\partial n} &= I [\delta(\theta - \alpha) - \delta(\theta + \alpha)]\end{aligned}$$

to obtain the following expression for $u(r, \theta)$:

$$u(r, \theta) = \frac{Ia}{2\pi\sigma} \log \left[\frac{a^2 + r^2 - 2ar \cos(\theta + \alpha)}{a^2 + r^2 - 2ar \cos(\theta - \alpha)} \right]$$

a) For the special case of axial stimulation $\alpha = \pi/2$, show that this solution reduces to

$$u(r, \theta) = \frac{Ia}{2\pi\sigma} \log \left[\frac{a^2 + r^2 + 2ar \sin \theta}{a^2 + r^2 - 2ar \sin \theta} \right]$$

It is also convenient to express the above solution in terms of an angle $\phi = \theta - \frac{\pi}{2}$ which is measured relative to the axis along which the two currents are being applied. In this case the solution becomes:

$$u_{\frac{\pi}{2}}(r, \phi) = \frac{aI}{2\pi\sigma} \log \left[\frac{a^2 + r^2 + 2ar \cos \phi}{a^2 + r^2 - 2ar \cos \phi} \right] \quad (1)$$

b) If the x axis is parallel to the stimulating axis show that the equipotential lines $u = A = \text{const}$ are circles centered on the x -axis of the form

$$(x + a\kappa)^2 + y^2 = \frac{4a^2\gamma}{(1 - \gamma)^2}, \text{ where } \gamma = e^A \text{ and } \kappa = \frac{1 + \gamma}{1 - \gamma}$$

Interpret and sketch the cases $A > 0$, $A = 0$, and $A < 0$.

c) If the x axis is parallel to the stimulating axis show that a given point (x, y) within the circle will find itself on the level set that passes through the points (X, Y) on the boundary of the circle $r = a$, where

$$X = \frac{2a^2x}{(x^2 + y^2 + a^2)}, \text{ and } Y = \pm\sqrt{a^2 - X^2} \quad (2)$$

4 The Sensitivity Theorem

4.1 Measurement assuming a perturbation to the uniform state

1) **Measurements:** Assume that the conductivity $\hat{\sigma}$ can be represented as a perturbation $\Delta\sigma$ to the uniform state σ , i.e., $\hat{\sigma} = \sigma + \Delta\sigma$

$$\begin{aligned} \nabla \cdot (\hat{\sigma} \nabla v_1) &= 0 \text{ on } \Omega = \{(r, \theta) : r \leq a, 0 \leq \theta < 2\pi\} \\ \hat{\sigma} \frac{\partial v_1}{\partial n} &= I \left[\delta\left(\theta - \frac{\pi}{2}\right) - \delta\left(\theta + \frac{\pi}{2}\right) \right] \end{aligned}$$

Now assume that the voltage is measured at two points along the boundary $(r, \theta) = (a, 0)$ and $(r, \theta) = (a, \frac{\pi}{8})$ say, and that the impedance difference is given by

$$\hat{z}_0 = \frac{1}{I} \left(v_1(a, \frac{\pi}{8}) - v_1(a, 0) \right) = \frac{\Delta v_1}{I}$$

2) **Nominal field:** Assume a uniform conductivity $\sigma = \text{const}$ and define the corresponding potential u_1 as the solution to the boundary value problem

$$\begin{aligned} \nabla \cdot (\sigma \nabla u_1) &= 0 \text{ on } \Omega = \{(r, \theta) : r \leq a, 0 \leq \theta < 2\pi\} \\ \sigma \frac{\partial u_1}{\partial n} &= I \left[\delta\left(\theta - \frac{\pi}{2}\right) - \delta\left(\theta + \frac{\pi}{2}\right) \right] \end{aligned}$$

Now evaluate the voltage is measured at the two points along the boundary $(r, \theta) = (a, 0)$ and $(r, \theta) = (a, \frac{\pi}{8})$, and the corresponding impedance difference is given by

$$z_0 = \frac{1}{I} \left(u_1(a, \frac{\pi}{8}) - u_1(a, 0) \right) = \frac{\Delta u_1}{I}$$

3) **Reciprocity field:** Assume a conductivity $\hat{\sigma}$ and define the potential v_0 as the solution to the boundary value problem

$$\begin{aligned} \nabla \cdot (\hat{\sigma} \nabla v_0) &= 0 \text{ on } \Omega = \{(r, \theta) : r \leq a, 0 \leq \theta < 2\pi\} \\ \hat{\sigma} \frac{\partial v_0}{\partial n} &= I \left[\delta(\theta - \frac{\pi}{8}) - \delta(\theta) \right] \end{aligned}$$

in which current is assumed to be applied to the boundary points at which the voltage measurements are taken.

Now use the divergence theorem to show that for given functions g, ψ , and ϕ

$$\int_{\Omega} \psi \nabla \cdot (g \nabla \phi) dA = \int_{\partial \Omega} \psi g \frac{\partial \phi}{\partial n} dl + \int_{\Omega} g \nabla \psi \cdot \nabla \phi dA \quad (3)$$

Now let $\psi = v_0, g = \sigma$ and $\phi = u_1$ in (3) and subtract the result from (3) with $\psi = u_1, g = \hat{\sigma}$ and $\phi = v_0$ to obtain an expression for the difference between Δu_1 and $v_0(a, \frac{\pi}{2}) - v_0(a, -\frac{\pi}{2})$ in terms of the integral $\int_{\Omega} \Delta \sigma \nabla u_1 \cdot \nabla v_0 dA$. Similarly, let $\psi = v_0, g = \hat{\sigma}$ and $\phi = v_1$ in (3) and subtract the result from (3) with $\psi = v_1, g = \hat{\sigma}$ and $\phi = v_0$ to obtain an expression for the difference $v_0(a, \frac{\pi}{2}) - v_0(a, -\frac{\pi}{2})$ in terms of Δv_1 .

Finally, derive the so-called sensitivity theorem

$$\hat{z}_0 - z_0 = \frac{1}{I} (\Delta v_1 - \Delta u_1) = -\frac{1}{I^2} \int_{\Omega} \Delta \sigma \nabla u_1 \cdot \nabla v_0 dA \quad (4)$$

Since we do not know $\hat{\sigma}$ it is impossible to determine ∇v_0 in (4). In this case we assume that $\hat{\sigma} = \sigma + \Delta \sigma$ where $\sigma \gg \Delta \sigma$ so that the change in conductivity $\Delta \sigma$ represents a small perturbation to the uniform state σ . Thus $v_0(\hat{\sigma}) = v_0(\sigma) + \Delta \sigma \frac{\delta v_0}{\delta \sigma} + O(\Delta \sigma^2) = u_0(\sigma) + \Delta \sigma \frac{\delta u_0}{\delta \sigma} + \dots$ where u_0 is the field obtained with a uniform conductivity σ and in which currents are applied at the same points along the boundary as for v_0 (i.e., $\sigma \frac{\partial u_0}{\partial n} = I [\delta(\theta - \frac{\pi}{8}) - \delta(\theta)]$). In this case the sensitivity theorem assumes the following approximate form

$$\hat{z}_0 - z_0 = \frac{1}{I} (\Delta v_1 - \Delta u_1) \simeq -\frac{1}{I^2} \int_{\Omega} \Delta \sigma \nabla u_1 \cdot \nabla u_0 dA + O(\Delta \sigma^2) \quad (5)$$

4.1.1 Discretization, level sets and the back-projection theorem

Firstly break the domain Ω into N elemental regions Ω_i on each of which $\Delta \sigma$ is assumed to be constant, i.e. $\Omega = \bigcup_{i=1}^N \Omega_i$. Assuming M measurements the approximate sensitivity theorem can be reduced to the form

$$S \Delta \sigma = \Delta z$$

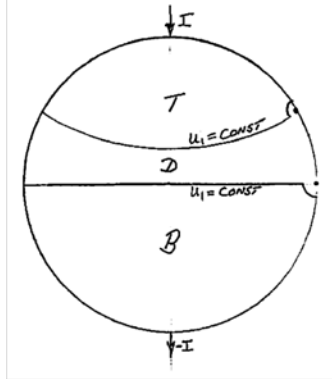


Figure 1:

in which S is the sensitivity matrix whose elements are defined by $S_{mi} = \int_{\Omega_i} \nabla u_1 \cdot \nabla u_0^m dA$ where u_0^m is the field associated with currents located at the m -th measurement pair. This under-determined system could be solved by least squares $S^T S \Delta \sigma = S^T \Delta z$, but $S^T S$ a very poorly conditioned matrix.

Back Projection:

1. One very special case results from the situation in which the boundaries of the subdomains Ω_i are defined by the level sets of the homogeneous source field u_1 (see figure 1). Now use the divergence theorem to prove the following results:

Claim 1: $\int_T \nabla u_1 \cdot \nabla u_0 dA = 0 = \int_B \nabla u_1 \cdot \nabla u_0 dA$

Claim 2: $\int_D \nabla u_1 \cdot \nabla u_0 dA = \frac{I}{\sigma} (u_1(a, \frac{\pi}{8}) - u_1(a, 0)) = \frac{I}{\sigma} \Delta u_1$

Claim 3: $\hat{z}_0 - z_0 = \frac{1}{I} (\Delta v_1 - \Delta u_1) \simeq -\frac{1}{I^2} \int_{\Omega} \Delta \sigma \nabla u_1 \cdot \nabla u_0 dA = -\frac{\Delta \sigma_D}{I \sigma} \Delta u_1$

Now combine these results to establish the back-projection formula for the “element” D :

$$\Delta \sigma_D = -\sigma \left(\frac{\Delta v_1 - \Delta u_1}{\Delta u_1} \right)$$

A code to implement the backprojection algorithm:

1. (a) Download the data file *Z.dat* from my web site: www.math.ubc.ca/~peirce. under math 406. This file contains the sequence of “measured data” which were generated using a finite element program. The files contains 128 = 16 × 8 data points which represent the voltage differences for 16 terminals uniformly distributed along the boundary of the circle. By symmetry, we can only obtain 8 independent axially distributed stimulating pairs $k - k'$. For each of these 8 axial orientations, we have 16 measurements as shown in the figure 2below.

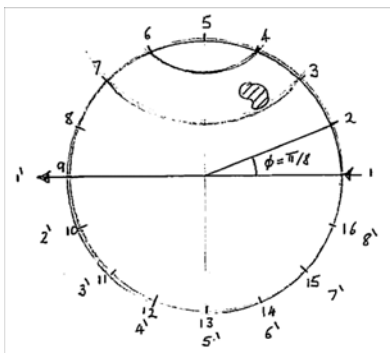


Figure 2:

The data in $Z.dat$ are organized as follows. The first of 16 elements correspond to the first axial pair located along the x axis i.e. $\phi = 0$. Thus Z_1 = the voltage difference between the stimulating axis T_1 and the next terminal marked T_2 . The subsequent 15 elements represent the voltage differences:

$$Z_k = u_{k+1} - u_k \text{ for } k = 1, \dots, 16$$

The next 16 elements in $Z.dat$ are those obtained for the same stimulating experiment but with the stimulating axis along $\phi = \frac{2\pi}{16}$. In this case the same convention was used to number the terminals starting with $k = 1$ for the stimulating axis. For the second axial pair the elements of $Z.dat$ are:

$$Z_{16+k} = u_{k+1} - u_k \text{ for } k = 1, \dots, 16$$

The remainder of the data in the file correspond to the voltage pairs for the stimulating axes oriented along $\phi = n \left(\frac{2\pi}{16} \right)$, $n = 2, \dots, 7$.

You will notice that the voltage differences that involve one of the terminals along the stimulating axis are infinite at the boundary. These have been set to some small number $\epsilon = 10^{-9}$ so that they have little effect on the data. When you code in the uniform solution given in (1) you should allocate the same values to these so that these voltages have no contribution.

- (b) Now implement the back projection procedure described in class. Divide the circle Ω_a with $a = 10$ into “pixels” (it may help to use polar coordinates). The for each pixel use the formulae (2) to locate the point (X, Y) on the boundary of the circle at which the level set intersects the boundary. This enables you to trap the pixel between two terminals so that the $\Delta\sigma$ that applies at the pixel can be determined using the voltage measurements for those two terminals according to the formula:

$$\Delta\sigma = -\sigma \left(\frac{\Delta v_1}{\Delta u_1} - 1 \right)$$

where Δv_1 are the voltage measurements that come from the experiment (i.e. $Z.dat$) and Δu_1 are the corresponding voltage differences calculated using the uniform solution (you will have to construct a similar vector to $Z.dat$ perhaps Zh which contains the voltage differences for a homogeneous medium). Since the medium was homogeneous only 16 such measurements for *one* axial pair are needed since the results for the other 7 are identical by symmetry. Accumulate all the $\Delta\sigma$ for all the measurements over all pixels and then plot the contours of the accumulated $\sigma(x, y)$ values over all pixels in the circle. This should, if you have coded the algorithm correctly, yield an image of where the anomaly is.